will not. However, Eq. (4) can be written in first order form as

$$\begin{bmatrix} c & m \\ m & 0 \end{bmatrix}\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix}+\begin{bmatrix} k & 0 \\ 0 & -m \end{bmatrix}\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}=0 \qquad (6)$$

Solutions can be found of the form

$$\begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}=\begin{Bmatrix} \phi_j \\ \phi_j\lambda_j \end{Bmatrix}e^{-\lambda_j t}$$

where λ_j and ϕ_j are the complex eigenvalues and eigenvectors from the $2n$ eigenproblem

$$\left(\begin{bmatrix} c & m \\ m & 0 \end{bmatrix}^{-1}\begin{bmatrix} k & 0 \\ 0 & -m \end{bmatrix}-\lambda_j\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\right)\begin{Bmatrix} \phi_j \\ \phi_j\ \lambda_j \end{Bmatrix}=0 \qquad (7)$$

as discussed in Ref. 1. Solution of this eigenproblem results in a set of coupled eigenvalues and eigenvectors which in the case of light damping may be written in matrix form as

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \phi & \phi^* \\ \phi\lambda & \phi^*\lambda^* \end{bmatrix}$$

Here λ denotes a diagonal matrix of complex eigenvalues and ϕ denotes the corresponding complex modal matrix. The asterisk denotes the complex conjugate. Since both of the square matrices in Eq. (6) are symmetric, the matrix

$$\begin{bmatrix} \phi^T & \lambda\phi^T \\ \phi^{*T} & \lambda^*\phi^{*T} \end{bmatrix}\begin{bmatrix} c & m \\ m & 0 \end{bmatrix}\begin{bmatrix} \phi & \phi^* \\ \phi\lambda & \phi^*\lambda^* \end{bmatrix}=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

is diagonal. In particular then

$$A_{12}=\phi^T c\phi^*+\phi^T m\phi^*\lambda^*+\lambda\phi^T m\phi^*=0 \qquad (8)$$

At this point, a perturbation analysis can be formulated by writing ϕ and ϕ^* in the form

$$\phi=\phi_R+\delta\phi_R+i\delta\phi_I \qquad (9a)$$

$$\phi^*=\phi_R+\delta\phi_R-i\delta\phi_I \qquad (9b)$$

where the complex eigenvectors ϕ are expressed as the sum of the hypothetical undamped eigenvectors ϕ_R which are real, and the relatively small complex perturbation terms $\delta\phi_R+i\delta\phi_I$. Substitution of Eq. (9) into Eq. (8) and the use of $\lambda_j=\sigma_j+i\omega_j$ leads to

$$\begin{aligned}
&\phi_j{}^T c\phi_k{}^*+\phi_j{}^T m\phi_k{}^*\lambda_k{}^*+\lambda_j\phi_j{}^T m\phi_k{}^*= \\
&\quad \phi_{R_j}{}^T c\phi_{R_k}+(\phi_{R_j}{}^T c\delta\phi_{R_k}+\delta\phi_{R_j}{}^T c\phi_{R_k}) \\
&-i(\phi_{R_j}{}^T c\delta\phi_{I_k}-\delta\phi_{I_j}{}^T c\phi_{R_k})+(\delta\phi_{R_j}{}^T c\delta\phi_{R_k}+\delta\phi_{I_j}{}^T c\delta\phi_{I_k}) \\
&-i(\delta\phi_{R_j}{}^T c\delta\phi_{I_k}-\delta\phi_{I_j}{}^T c\delta\phi_{R_k})+[(\sigma_j+\sigma_k)+i(\omega_j-\omega_k)]\times \\
&[\phi_{R_j}{}^T m\phi_{R_k}+(\phi_{R_j}{}^T m\delta\phi_{R_k}+\delta\phi_{R_j}{}^T m\phi_{R_k}) \\
&-i(\phi_{R_j}{}^T m\delta\phi_{I_k}-\delta\phi_{I_j} m\phi_{R_k})+(\delta\phi_{R_j}{}^T m\delta\phi_{R_k}+\delta\phi_{I_j}{}^T m\delta\phi_{I_k}) \\
&\qquad\qquad -i(\delta\phi_{R_j}{}^T m\delta\phi_{I_k}-\delta\phi_{I_j}{}^T m\delta\phi_{R_k})]=0
\end{aligned}$$

In the preceding equation, ω, m, and ϕ_R are treated as zeroth-order terms while σ, c, $\delta\phi_R$, and $\delta\phi_I$ are treated as first-order terms. Zeroth- and first-order product terms, real and imaginary, are then separately equated to zero. This leads to the following equations:

$$(\omega_j-\omega_k)\phi_{R_j}{}^T m\phi_{R_k}=0 \qquad (10a)$$

$$\phi_{R_j}{}^T c\phi_{R_k}=-(\sigma_j+\sigma_k)\phi_{R_j}{}^T m\phi_{R_k}: \quad j=k \qquad (10b)$$

$$\phi_{R_j}{}^T c\phi_{R_k}=-(\omega_j-\omega_k)(\phi_{R_j}{}^T m\delta\phi_{I_k}-\phi_{R_j}{}^T m\phi_{R_k}): \quad j\neq k \qquad (10c)$$

Equation (10a) is simply a statement of the orthogonality of ϕ_R with respect to m. Equations (10b and c) become

$$C_{jj}=-2\sigma_j M_{jj} \qquad (11a)$$

$$\begin{aligned} C_{jk}&=-(\omega_j-\omega_k)(\phi_{R_j}{}^T m\delta\phi_{I_k}-\delta\phi_{I_j}{}^T m\phi_{R_k}) \\ &=\omega_j\delta\phi_{I_j}{}^T m\phi_{R_k}+\omega_k\phi_{R_j}{}^T m\delta\phi_{I_k}: \quad j\neq k \end{aligned} \qquad (11b)$$

which are expressions for the diagonal and off-diagonal elements of C. Since $\sigma_j=-\zeta_j\omega_{0_j}$, it follows that the diagonal elements of C correspond to modal damping as in the case of proportional damping. The off-diagonal elements of C can be calculated provided that $\delta\phi_I$ can be measured.

One possible way to measure $\delta\phi_I$ suggests itself from the co-quad technique currently employed to measure modal frequencies and damping.[4,5] The acceleration response of a given point on a structure excited in one of its "normal" modes is separated into coincident and quadrature components (co and quad) with respect to either the driving force or some other response point on the structure. These components are 90° out of phase with each other, the coincident part defined to be in phase with the reference function. They are thus representable as real and imaginary components in the complex plane as illustrated by Fig. 2. In the case where the driving force is taken to be the reference function, the real part of the modal displacement corresponds to the quadrature component of response since response is nearly 90° out of phase with the excitation at resonance. The coincident response data are not fully utilized. The method proposed herein could make use of this information to construct the off-diagonal terms of the modal damping matrix, provided that sufficiently "pure" modes are obtainable.

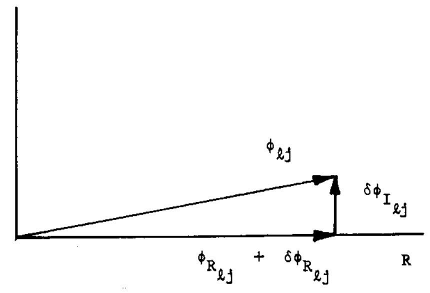

Fig. 2 Vectorial representation of the lth element of the jth eigenvector in the complex plane.

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Control Theory Formulation for Nonlinear Elastic Analysis of Trusses

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Introduction

METHODS for the analysis of structures for which the application of the superposition principles is not justified normally envisage the solution of a set of nonlinear equations (either algebraic or differential).[1,2,4] It is desirable to develop methods of nonlinear structural analysis which result in a set of nonlinear equations with a smaller number of basic unknowns. To do this, herein, the structure is visualized as a

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controlled system and the technique of "state space analysis" is used.[5,6] The method is applicable to statically indeterminate trusses having conservative nonlinear elements.

Description of the Method

The structure is partitioned into a series of statically determinate substructures, herein called, "stages," such that each substructure is coupled to two adjacent substructures. Let there be a total of M stages. The sets of forces acting on a typical mth stage are: 1) the set $X^{(m-1)}$ of interacting forces $x_i^{(m-1)}$, $i = 1, 2, \ldots, N_{m-1}$ between $(m-1)$th stage and mth stage; 2) the set $X^{(m)}$ of interacting forces $x_i^{(m)}$, $i = 1, 2, \ldots, N_m$ between mth stage and $(m+1)$th stage; and 3) the set $P^{(m)}$ of externally applied forces $p_i^{(m)}$, $i = 1,2,\ldots,L_m$ for the stage. $X^{(m)}$ is considered to be the set of state variables corresponding to mth stage and $P^{(m)}$, $m = 1,2,\ldots, M$ will be the inputs to the system. Properties of the members (area of cross section, length of the member and Young's modulus for the material of the member) forming the mth stage will be the elements of the set $U^{(m)}$ of control parameters of the mth stage.

Using equilibrium and compatibility conditions it is possible to write the "state equations" for the system. Whereas for the case of linear structures it is readily possible to write explicitly the state transformation equations for $X^{(m)}$ in terms of $X^{(m-2)}$, $X^{(m-1)}$, $P^{(m-1)}$, $P^{(m)}$, $U^{(m-1)}$ and $U^{(m)}$, it is not always possible to write explicit equations for nonlinear structures. But, in such circumstances, a sequence of operations explained below will serve the same purpose as the state equations and will be used to evaluate $X^{(m)}$ using the values of $X^{(m-2)}$, $X^{(m-1)}$, $P^{(m-1)}$, $P^{(m)}$, $U^{(m-1)}$, and $U^{(m)}$. The members of the mth stage will be divided into the following two mutually exclusive sets; 1) the set $J_1^{(m)}$ of members for which the internal forces can be evaluated using statics when $X^{(m-1)}$ and $P^{(m)}$ are known; and 2) the set $J_2^{(m)}$ of members for which the internal forces can be evaluated using statics, only when $X^{(m)}$ is also known in addition to $X^{(m-1)}$ and $P^{(m)}$. In the case of planar trusses the elements $x_{N_m-2}^{(m)}, x_{N_m-1}^{(m)}$ and $x_{N_m}^{(m)}$ of $X^{(m)}$ will be denoted as dependent elements of $X^{(m)}$ since they can always be evaluated using statics in terms of the other independent elements of $X^{(m)}$ and the elements of $X^{(m-1)}$ and $P^{(m)}$. The $(m-1)$th stage is restrained to have zero displacements in the directions of dependent components of $X^{(m-1)}$ and the displacements $\delta_i^{(m-1,m-1)}$, $i = 1,2,\ldots, N_{m-1}-3$ corresponding to the independent components of $X^{(m-1)}$ are calculated for the $(m-1)$th stage in terms of the known $X^{(m-2)}$, $X^{(m-1)}$, $P^{(m-1)}$ and $U^{(m-1)}$. The internal forces $F_j^{(m)}$ and total elongations $\Delta_j^{(m)}$ for $j \in J_1^{(m)}$ are evaluated using known values of the elements of $X^{(m-1)}$, $P^{(m)}$, and $U^{(m)}$. Now the mth stage is constrained to have zero displacements in the directions of dependent components of $X^{(m-1)}$ and the displacements $\delta_i^{(m,m-1)}$, $i = 1,2,\ldots, N_{m-1}-3$ corresponding to the independent components of $X^{(m-1)}$ are determined for mth stage in terms of the known $\Delta_j^{(m)}$'s, $j \in J_1^{(m)}$ and unknown $\Delta_j^{(m)}$'s, $j \in J_2^{(m)}$. For compatibility of displacements,

$$\delta_i^{(m,m-1)} = \delta_i^{(m-1,m-1)} \qquad i = 1,2,\ldots,N_{m-1}-3 \qquad (1)$$

Eq. (1) gives a set of linear equations for the unknowns $\Delta_j^{(m)}$'s, $j \in J_2^{(m)}$. This set is solved to get the values for $\Delta_j^{(m)}$'s, $j \in J_2^{(m)}$. Knowing the total deformations the forces $F_j^{(m)}$'s, $j \in J_2^{(m)}$ are determined using the force-deformation relationship for individual members. But $F_j^{(m)}$'s, $j \in J_2^{(m)}$ are also given by

$$F_j^{(m)} = \sum_{i=1}^{N_m} a_{ij}^{(m,m)} x_i^{(m)} + \sum_{i=1}^{N_{m-1}} a_{ij}^{(m,m-1)} x_i^{(m-1)} + \sum_{i=1}^{L_m} b_{ij}^{(m)} p_i^{(m)} \qquad j \in J_2^{(m)} \qquad (2)$$

where $a_{ij}^{(m,m)}$, $a_{ij}^{(m,m-1)}$, and $b_{ij}^{(m)}$ are constant coefficients determined using equilibrium conditions. Equations (2) represent a set of linear equations in the unknowns $x_i^{(m)}$, $i = 1,2,\ldots, N_m$. Solving this set, $X^{(m)}$ is evaluated. Thus it is possible to calculate $X^{(m)}$ when $X^{(m-2)}$, $X^{(m-1)}$, $P^{(m-1)}$, $P^{(m)}$, $U^{(m-1)}$, and $U^{(m)}$ are known.

Because of the availability of the sequence of operations equivalent to the state equations, the unknowns in the problem are the interacting forces $X^{(0)}$ and $X^{(1)}$ acting on the first stage and thus these forces represent "the state of the system." To determine $X^{(0)}$ and $X^{(1)}$, any method, such as the Newton-Raphson technique, which is used to solve a set of nonlinear algebraic equations, can be used. Usually, the values for elements of $X^{(0)}$ and the independent elements of $X^{(1)}$ are assumed. (In the case of open loop structures, where the first stage and the last stage have only one adjacent stage, the $X^{(0)}$ is identically equal to zero.) Using statics, the dependent components of $X^{(1)}$ are determined. Performing the set of operations outlined previously, in succession, the elements of $X^{(m)}$, $m = 2,3,\ldots, M$ are determined. If the values of elements of $X^{(M)}$ satisfy the known boundary conditions for last stage, the assumed values for $X^{(0)}$ and $X^{(1)}$ are correct. (The known boundary conditions for Mth stage are; for open loop structures, $x_i^M = 0$; $i = 1,2,\ldots, N_M$ and for closed loop structures, where the last stage is the same as the first stage, $x_i^{(1)} = x_i^{(M)}$; $i = 1,2,\ldots, N_M$ and $x_i^{(0)} = x_i^{(M-1)}$, $i = 1,2,\ldots, N_{M-1}$.) If the boundary conditions are not satisfied, the assumed values of $X^{(0)}$ and $X^{(1)}$ are corrected in a suitable manner till the boundary conditions are satisfied.

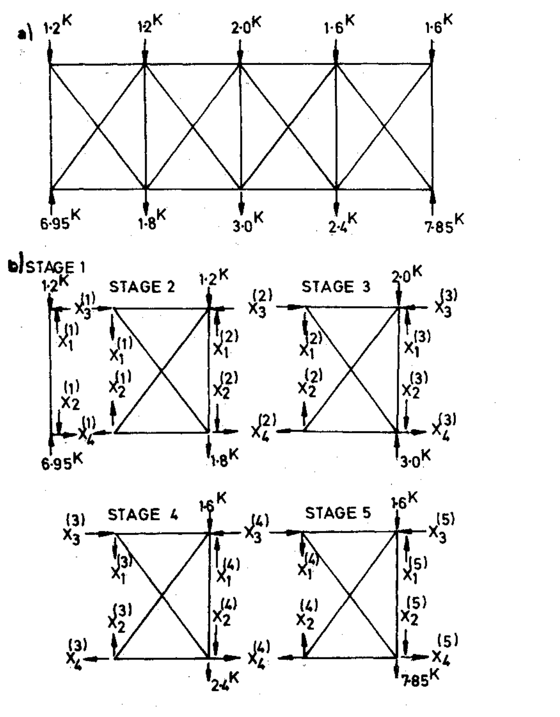

Fig. 1 a) Nonlinear truss and loading on it; b) various stages and loading on them.

Numerical Example

The truss as shown in Fig. 1 is analyzed using the preceding method. The loading on the truss is also shown in Fig. 1. The truss elements are assumed to have the following nonlinear force-deformation relations as suggested by Goldberg and Richards.[2,4]

$$F_j = \frac{A_j E_j \Delta_j}{l_j} \frac{1}{\left\{1+\left|\frac{A_j E_j \Delta_j}{l_j F_{0j}}\right|^{n_{2j}}\right\}^{1/n_{2j}}} \qquad (3)$$

and

$$\Delta_j = F_j l_j/A_j E_j + \tfrac{3}{7}(S_{0j} l_j/A_j E_j)(F_j/S_{0j})^{n_{1j}} \qquad (4)$$

The properties of truss elements are given in Table 1. The stages of the truss are shown in Fig. 1b. For this structure,

$x_i^{(0)} = 0$, $i = 1$–4, and also $x_i^{(5)} = 0$, $i = 1, 2, 3, 4$. The only unknown is $x_1^{(1)}$. (It is to be noted that the number of unknowns is only one irrespective of the number of stages unlike in other available methods.) To determine $x_1^{(1)}$, unidimensional search procedure was used. The results are given in Table 2. Using the values of interacting forces given in Table 2, the internal forces in the members can be calculated.

Table 1 Properties of Elements

	Horizontal members	Vertical members	Diagonal members
1) Area of cross section, sq. in.	00.25	00.20	00.20
2) Length of the member, in.	15.00	20.00	25.00
3) Young's modulus in ksi	10^4	10^4	10^4
4) Characteristic force F_0, kips	11.65	08.54	08.54
5) Characteristic force S_0, kips	10.13	08.10	08.10
6) Parameter n_1	05.00	05.00	05.00
7) Parameter n_2	07.00	07.00	07.00

Table 2 State variables for the example problem

	$X^{(1)}$, kips	$X^{(2)}$, kips	$X^{(3)}$, kips	$X^{(4)}$, kips	$X^{(5)}$, kips
1)	−2.2827	−1.5534	0.9148	3.7984	−0.0000
2)	+3.4673	+1.1966	−1.3352	−2.4516	0.0000
3)	0.0000	4.3125	6.3750	4.6875	0.0000
4)	0.0000	4.3125	6.3750	4.6875	0.0000

Conclusions

A method of analysing trusses made of nonlinear elastic materials was presented. Considerable reduction in the number of unknowns to be determined is possible for structures which can be partitioned into a series of substructures with a consequent reduction in computer storage capacity requirements. Possible disadvantages with the method may be that the round-off errors may tend to be multiplied as the calculations are carried out from one stage to next stage.

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Annular Plate with Supporting Edge Beams

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Introduction

ANNULAR plates are used extensively in some aerospace structures. Here, an annular plate framing into simply supported edge beams and loaded by a concentrated load is considered as shown in Fig. 1. The deflection of the plate, w, is given within the theory of thin elastic plates by the expression

$$\nabla^4 w = q/D \tag{1}$$

where ∇^2 is the Laplacian operator applied twice, q is the intensity of the lateral loading, and D is the flexural rigidity of the plate.

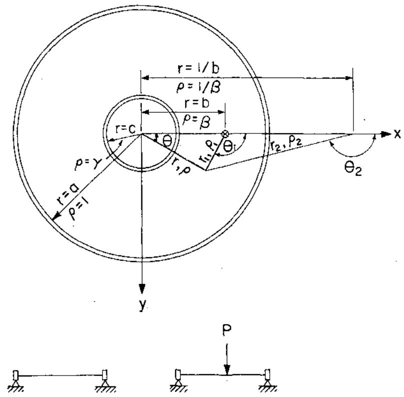

Fig. 1 Annular plate with supporting edge beams.

Analysis

The general solution of Eq. (1) is expressed as

$$w = w_0 + w_1 \tag{2}$$

where w_0 is the particular solution and w_1 the homogeneous solution. The particular solution for the annular plate problem subjected to a concentrated force is taken in the form[1]

$$w_0 = (\rho a^2/8\pi D)\rho_1^2 \log(\rho_1/\beta\rho_2) \tag{3}$$

where the radial coordinate r is nondimensionalized by setting

$$\rho = r/a \tag{4}$$

Equation (3) represents the proper singularity term which must be present in the solution due to the concentrated force.[2]

The homogeneous solution is[3]

$$w_1 = \frac{\rho a^2}{8\pi D} \sum_{n=0}^{\infty} R_n \cos n\theta \tag{5}$$

where

$$R_0 = A_0 + B_0\rho^2 + C_0 \log\rho + D_0\rho^2 \log\rho \tag{6}$$

$$R_1 = A_1\rho + B_1\rho^3 + C_1\rho^{-1} + D_1\rho\log\rho \tag{7}$$

$$R_n = A_n\rho^n + B_n\rho^{-n} + C_n\rho^{n+2} + D_n\rho^{-n+2} \quad n \geq 2 \tag{8}$$

where A, B, C, and D are constants.

The deflection function is required to satisfy the following boundary conditions[4]

$$w = 0 \quad \text{on } \rho = 1 \tag{9}$$

$$w = 0 \quad \text{on } \rho = \gamma \tag{10}$$

$$M_\rho = (D/a^2)(\partial/\partial\rho)[\kappa_1 w - \lambda_1(\partial^2 w/\partial\theta^2)] \quad \text{on } \rho = 1 \tag{11}$$

$$M_\rho = -(D/a^2)(\partial/\partial\rho)[\kappa_2 w - \lambda_2(\partial^2 w/\partial\theta^2)] \quad \text{on } \rho = \gamma \tag{12}$$

Here,

$$\kappa_1 = K_1/aD; \quad \lambda_1 = L_1/aD \tag{13}$$

$$\kappa_2 = K_2/\gamma aD; \quad \lambda_2 = L_2/\gamma aD \tag{14}$$

where K and L are the flexural and torsional rigidities of the edge

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